ALL SAINTS' COLLEGE

Mathematics
Specialist
Test 4 2019

Integration Techniques & Applications of Integral Calculus

NAME: SOLUTIONS

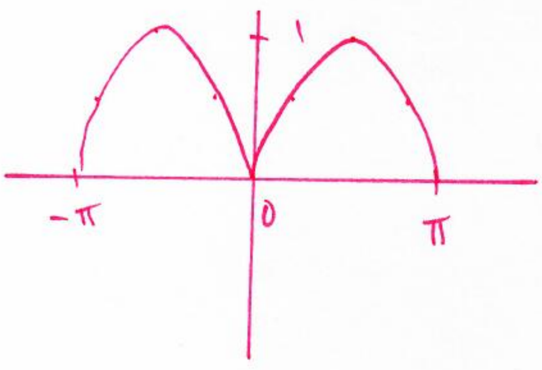
TEACHER: Mrs Da Cruz

Resource Free Section

25 marks
25 minutes

Question 1 **[3, 2, 5, 4, 5, 2, 4 = 25 marks]**

(a) $\int_{-\pi}^{\pi}|\sin x|dx$ (Hint: sketch the function first.) [3]



$$\int_{-\pi}^{\pi}|\sin x|\,dx = 2\int_{0}^{\pi}\sin(x)\,dx \checkmark$$

$$= 2\left[-\cos(x)\right]_0^{\pi} \checkmark$$

$$= 2\left[-\cos\pi - (-\cos 0)\right]$$

$$= 2\left[--1 - -1\right]$$

$$= 2[2]$$

$$= 4 \checkmark$$

(b) Given that $\frac{d}{dx}[xe^x - e^x] = xe^x$, find $\int xe^x + x\,dx$. [2]

$$= \underbrace{xe^x - e^x}_{\checkmark} + \underbrace{\frac{x^2}{2}}_{\checkmark} + c$$

(c) Use the substitution $u = \ln x$ to determine $\int \frac{\sqrt{\ln x} + ln\sqrt{x}}{x}\,dx$. [~~5~~] 6

$u = \ln x \Rightarrow x = e^u$ ✓ or $\ln x^{1/2} = \frac{1}{2}\ln x = \frac{u}{2}$.

$\frac{du}{dx} = \frac{1}{x}$ ✓

$$\therefore \int \frac{\sqrt{\ln x} + \ln\sqrt{x}}{x}\,dx = \int \sqrt{u} + \ln\sqrt{e^u}\,du \quad ✓$$

$$= \int u^{1/2} + \ln e^{u/2}\,du$$

$$= \int u^{1/2} + \frac{u}{2}\,du \quad ✓$$

$$= \frac{2}{3}u^{3/2} + \frac{u^2}{4} + c \quad ✓$$

$$= \frac{2}{3}[\ln x]^{\frac{3}{2}} + \frac{[\ln x]^2}{4} + c \quad ✓$$

(c) $\int \sin^3 2t\,dt$ $= \int \sin 2t \,.\, \sin^2 2t \; dt$ [4]

$= \int \sin 2t \left[1 - \cos^2 2t\right] dt$ ✓

$= \int \sin 2t - \sin 2t \cos^2 2t \quad dt$ ✓

$= -\frac{\cos 2t}{2} + \frac{\cos^3 2t}{6} + c$ ✓ ✓

(d) Using partial fractions, find $\int \frac{1}{2x^2-x-6}\,dx$ [5]

$= \int \frac{1}{(2x+3)(x-2)} \, dx$ ✓

$\frac{1}{(2x+3)(x-2)} = \frac{A}{2x+3} + \frac{B}{x-2}$

$\therefore 1 = A(x-2) + B(2x+3)$

$x=2$: $1 = B(7)$

$B = 1/7$ ✓

$x = -\frac{3}{2}$: $1 = A\left(-\frac{3}{2} - 2\right) + 0$

$1 = A\left(-\frac{7}{2}\right)$

$A = -\frac{2}{7}$ ✓

$\int \frac{1}{(2x+3)(x-2)}\,dx = \int \frac{-2}{7(2x+3)} + \frac{1}{7(x-2)}\,dx$ ✓

$= \frac{1}{7} \int \frac{1}{x-2} - \frac{2}{2x+3}\,dx$

$= \frac{1}{7}\left[\ln|x-2| - \ln|2x+3|\right] + c$ ✓

$= 1/7 \ln\left|\frac{x-2}{2x+3}\right| + c$

(e) $\int \frac{e^{2x}}{3+2e^{2x}}\,dx$ $= \frac{1}{4}\int \frac{4e^{2x}}{3+2e^{2x}}\,dx$ [2]

$= \frac{1}{4}\ln\left(3+2e^{2x}\right) + c$ ✓✓

(f) $\int_0^1 \frac{1-x}{x+1}\,dx$ $(Hint: You\ could\ use\ \ u = x+1)$ [4]

$u = x+1 \Rightarrow x = u-1$

$\frac{du}{dx} = 1$

$x=1,\ u=2$

$x=0,\ u=1$ ✓

$\int_1^2 \frac{1-(u-1)}{u}\,du$ ✓

$= \int_1^2 \frac{2-u}{u}\,du$

$= \int_1^2 \frac{2}{u} - 1\,du$ ✓

$= \left[2\ln u - u\right]_1^2$

$= 2\ln 2 - 2 - [\cancel{2\ln 1}^{0} - 1]$

$= 2\ln 2 - 1$ ✓

Alternately:

$-\int_0^1 \frac{x-1}{x+1}\,dx$

$= -\int_0^1 \frac{x+1-2}{x+1}\,dx$ ✓

$= -\int_0^1 1 - \frac{2}{x+1}\,dx$ ✓

$= -\left[x - 2\ln(x+1)\right]_0^1$ ✓

$= -\left[1 - 2\ln 2 - (0 - \cancel{2\ln 1}^{0})\right]$

$= -\left[1 - 2\ln 2\right]$

$= 2\ln 2 - 1$ ✓

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Resource Rich Section

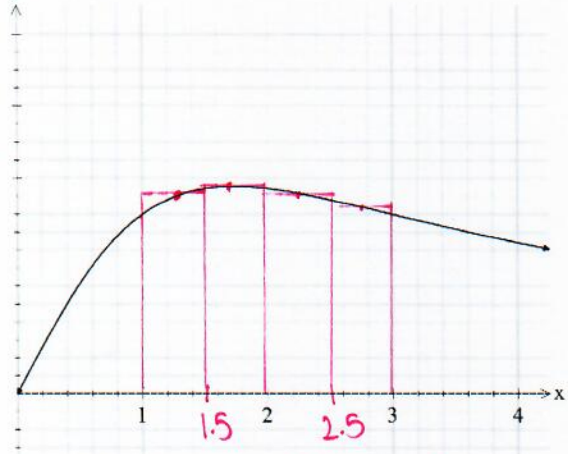
19 marks
25 minutes

One unfolded A4 page of notes, SCSA formulae booklet and ClassPad calculator permitted.
Show sufficient working for marks to awarded.

Question 2 **[2 marks]**

Consider the area under the curve $f(x) = \frac{2x}{3+x^2}$ between $x = 1$ and $x = 3$. Using four mid-point rectangles **approximate the area.**

using eActivity: Area $\approx$ 1.102 units^2



1.25 1.75 2.25 2.75 } midpoint x-values

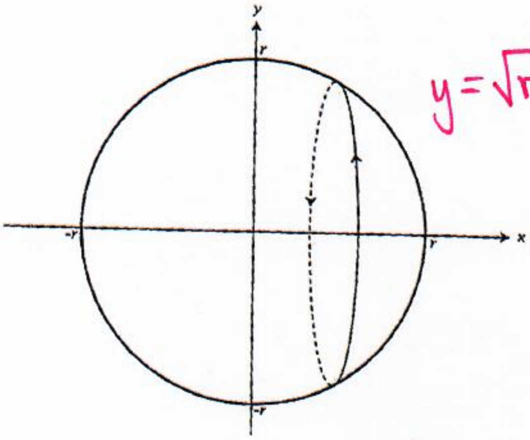
x	y
1.25	0.5479
1.75	0.5773
2.25	0.5581
2.75	0.5207

$\therefore A \approx 0.5(0.5479 + 0.5773 + 0.5581 + 0.5207)$

$\therefore A \approx 1.102$ units^2

Question 3 **[4 marks]**

Show that the volume of a sphere, $V = \frac{4\pi r^3}{3}$, may be generated by rotating the circle $x^2 + y^2 = r^2$ about the x-axis.



$y = \sqrt{r^2 - x^2}$

$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx \quad ✓$$

$$= 2\pi \int_{0}^{r} r^2 - x^2 \, dx$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \quad ✓$$

must include $*$

$$= 2\pi \left[r^3 - \frac{r^3}{3} - 0 + 0 \right] \quad ✓$$

$$= 2\pi \left[\frac{2r^3}{3} \right]$$

$$\therefore V = \frac{4\pi r^3}{3} \quad ✓$$

or

$$= \pi \int_{-r}^{r} r^2 - x^2 \, dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \quad ✓$$

✓ at least one of these

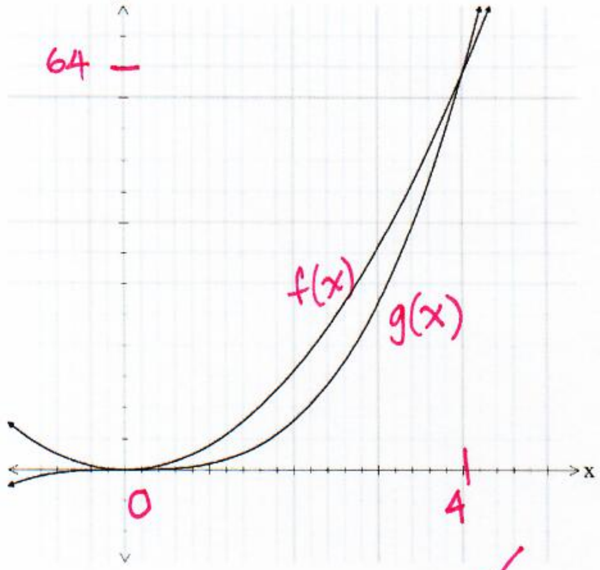
$$= \pi \left[r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3} \right) \right]$$

$$= \pi \left[r^3 \left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3} \right) \right]$$

$$\therefore V = \frac{4\pi r^3}{3} \quad ✓$$

Question 4 **[5 marks]**

Consider the region bounded by the curves $f(x) = 4x^2$ and $g(x) = x^3$ for $x \geq 0$. Determine the volume of the solid formed when the region is rotated about the y-axis.



$y = 4x^2$

$\frac{y}{4} = x^2$

$x = \sqrt{\frac{y}{4}}$

$x = \frac{y^{1/2}}{2}$

$y = x^3$

$y^{1/3} = x$ ✓

$$V = \pi \int_0^{64} \left(y^{1/3}\right)^2 - \left(\frac{y^{1/2}}{2}\right)^2 dy \quad ✓ ✓$$

$$= \pi \int_0^{64} y^{2/3} - \frac{y}{4}\, dy$$

$$= \pi \left[\frac{3}{5} y^{5/3} - \frac{1}{8} y^2\right]_0^{64} \quad ✓$$

* required.

$$= \pi \left[\frac{3}{5} 64^{5/3} - \frac{1}{8} 64^2 - 0\right]$$

$$\therefore V = \frac{512\pi}{5} \text{ units}^3 \quad ✓$$

Alternate: $V = 2\pi \int_0^4 x\left(4x^2 - x^3\right) dx$ ✓✓

$$= 2\pi \int_0^4 4x^3 - x^4\, dx$$

$$= 2\pi \left[x^4 - \frac{x^5}{5}\right]_0^4 \quad ✓$$

$$= 2\pi \left[\frac{256}{5}\right]$$

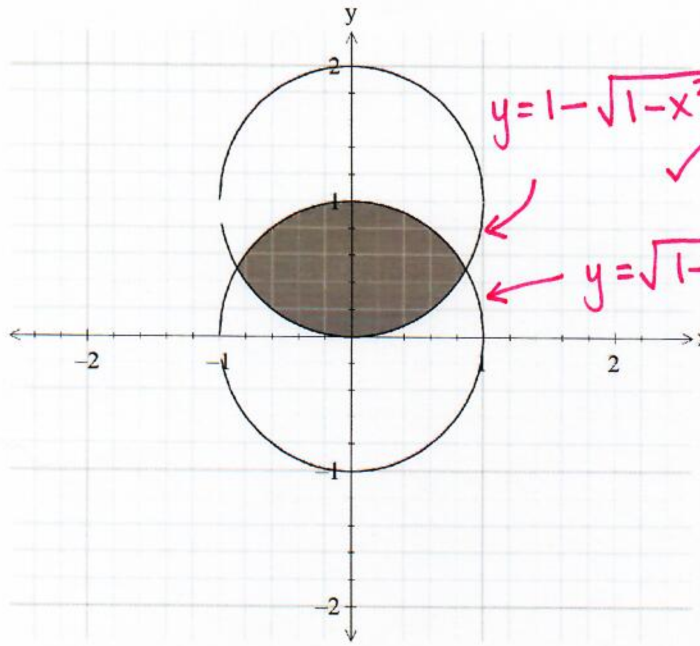
$$\therefore V = \frac{512\pi}{5} \text{ units}^3 \quad ✓$$

* must show integrated function for all 5 marks.

Question 5 **[8 marks]**

Find the area of the lens shape formed between two circles, $x^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$.

(Hints: You will need to first find the relevant semi-circle equations. Use the substitution $x = \sin\theta$)



$y = 1 - \sqrt{1-x^2}$ ✓

$y = \sqrt{1-x^2}$

Interc-epts at $\pm \frac{\sqrt{3}}{2}$ ✓

$x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

✓ $\begin{cases} x = 0 = \sin\theta \,;\ \theta = 0 \\ x = \frac{\sqrt{3}}{2} = \sin\theta \,;\ \theta = \frac{\pi}{3} \end{cases}$

$\sqrt{1-x^2}$

$= \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta}$

$= \cos\theta$. ✓

$$A = 2\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2} - \left(1 - \sqrt{1-x^2}\right) dx \ ✓$$

$$= 2\int_0^{\frac{\sqrt{3}}{2}} 2\sqrt{1-x^2} - 1 \ dx$$

$$= 2\int_0^{\frac{\pi}{3}} (2\cos\theta - 1)\cos\theta \, d\theta \ ✓$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\int_0^{\pi/3} 2\cos^2\theta - \cos\theta \ d\theta$$

$$= 2\int_0^{\pi/3} \cos 2\theta + 1 - \cos\theta \ d\theta$$

$$= 2\left[\frac{\sin 2\theta}{2} + \theta - \sin\theta\right]_0^{\pi/3} \ ✓$$

$$= 2\left[\frac{\sqrt{3}}{4} + \frac{\pi}{3} - \frac{2\sqrt{3}}{4}\right]$$

$$\therefore A = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \text{ Units} \ ✓$$